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A Linear Approximation for Chance-Constrained Programming

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Decision environments involve the need to solve problems with varying degrees of uncertainty as well as multiple, potentially conflicting objectives. Chance constraints consider the uncertainty encountered. Codes incorporating chance constraints into a mathematical programming model are not available on a widespread basis owing to the non-linear form of the chance constraints. Therefore, accurate linear approximations would be useful to analyse this class of problems with efficient linear codes. This paper presents an approximation formula for chance constraints which can be used in either the single- or multiple-objective case. The approximation presented will place a bound on the chance constraint at least as tight as the true non-linear form, thus overachieving the chance constraint at the expense of other constraints or objectives.

Key words: chance-constrained programming, goal programming, multiple-objective programming

INTRODUCTION

Consideration of risk is a major element in decision-making. Risk can be considered in a variety of ways. Chance-constrained programming can be applied to incorporate sampling information, where a measurable variance for coefficients is known.¹⁻³

Use of chance-constrained programming introduces a new requirement upon decision-makers. The prescribed level of risk must be selected. Risk generally conflicts with other objectives, such as maximization of profit or minimization of cost. In purely objective studies, with a clear single objective, risk could be incorporated in the objective function, as in stochastic programming with recourse. In managerial decision-making, risk is not always linearly commensurate with other objectives, such as profitability or cost. Utility provides a theoretical, totally ordered preference function, expected to be non-linear. Total expression of this utility function is difficult. Chance-constrained multiple-objective analyses provide a means of coping with, or developing, this trade-off. Use of chance constraints in decision problems requires balancing risk with other objectives. This implementational requirement has led to a number of recent multiobjective chance-constrained studies. Most of these utilized goal programming.⁴⁻⁷ Rakes and Reeves discussed extensions to more general multiobjective analysis.⁸

A major difficulty in using chance-constrained programming is the need for a non-linear algorithm. Two approaches have been developed, both capable of solving chance-constrained models. Rakes *et al.*⁶ used a piecewise linear goal-programming code. Lee and Olson⁷ presented a gradient code for goal programming models involving chance constraints. While both codes are functional, the separable code involves a significant amount of model manipulation to develop the piecewise linear functions, and both codes are long and not readily transferable to other computer systems. De *et al.*⁴ demonstrated how Naslund's⁹ zero-one linear approximation can be utilized in chance-constrained zero-one goal programming models. This paper extends that idea to continuous chance-constrained models, using a linear approximation that more closely reflects chance constraints than Naslund's approximation.

CHANCE-CONSTRAINED PROGRAMMING

Chance-constrained programming was developed as a means of describing constraints in the form of probability levels of attainment. Consideration of chance constraints allows the decision-maker to consider objectives in terms of their attainment probability. If α is a predetermined confidence level desired by the decision-maker, the implication is that a constraint will have a probability of satisfaction of α .

The probabilistic nature of this approach lends itself to multiobjective analysis. The selection of α can be a managerial decision, as there will be a trade-off required between other objectives and minimization of risk. One approach, stochastic programming with recourse, would be to develop the costs of failure for all chance constraints, and include these costs of failure, weighted by probability, in the objective function. However, this approach can easily involve more subjective assessment than setting the level of risk. In addition, the analysis would be far more involved. A multiobjective analysis would focus upon the trade-off between risk and cost or profit. This can be done parametrically, expediting a utility trade-off analysis, or pre-emptive goal-programming techniques can be used to obtain a solution to a prescribed list of objective attainments.

Chance constraints for stochastic functions based upon sampling information would often be normally distributed. Sampling information has long been used in business as a means of determining the expected value of functional coefficients in linearly constrained systems. The added use of variance information would provide a more complete analysis when the estimated coefficients are based upon some stochastic estimates of a_{ij} values. Some applications of this have been noted in the literature by Hogan *et al.*¹⁰ However, this added information is usually disregarded, primarily because there is not a widely available solution method.

FORM OF CHANCE CONSTRAINTS

Chance constraints use variance (variance-covariance) information. In constraint i :

$$\sum_{j=1}^n a_{ij}x_j \leq b_i.$$

If the parameters a_{ij} are based upon sampling or other stochastic information (and the estimates of a_{ij} are normally distributed), it is appropriate to consider the probability level of constraint satisfaction. For the case of independent a_{ij} ,

$$\sum_{j=1}^n a_{ij}x_j$$

represents the expected value of the function.

Let

$$\sum_{j=1}^n v_{ij}x_j^2$$

be the variance-covariance element of the a_{ij} s. The chance constraint would be:

$$\sum_{j=1}^n a_{ij}x_j + z \left[\sum_{j=1}^n v_{ij}x_j^2 \right]^{0.5} \leq b_i,$$

where z represents the normal density functional associated with the prescribed confidence level α . The constraint set is convex only if $\alpha \geq 0.5$. For greater than or equal relationships,

$$\sum_{j=1}^n a_{ij}x_j - z \left[\sum_{j=1}^n v_{ij}x_j^2 \right]^{0.5} \geq b_i.$$

The variance-covariance term acts as a penalty function. Figure 1 shows how the penalty works in a two-variable case for the chance constraint:

$$5X + 4Y + z[4X^2 + 1Y^2]^{0.5} = 10.$$

This procedure does not require a_{ij} elements to be independent. However, chance constraints must be independent of each other. If chance constraints are not independent, joint chance-constrained programming procedures would be required. If only parameter b_i is stochastic, the right-hand side is adjusted for the selected degree of probability and the standard deviation of b_i . The resulting penalty does not result in non-linearity, and linear programming codes can be used for solution (see Keown and Taylor¹³ as an example).

For stochastic a_{ij} , a useful linear approximation of the chance constraint can be developed, assuring that at least the prescribed probability of constraint satisfaction is obtained. The greatest

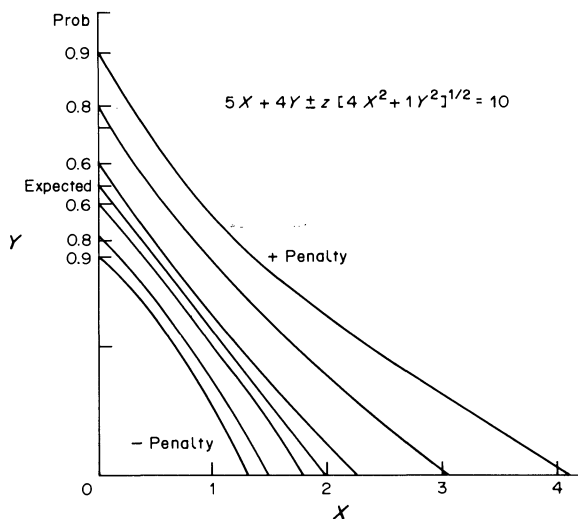


FIG. 1. Impact of probability level.

penalty occurs at the decision-variable extremes. A line connecting these extremes will have the property that the approximation will always impose at least as severe a penalty as the chance constraint. For the general chance constraint:

$$fX + gY + z[uX^2 + vY^2]^{0.5} \leq b,$$

where u is the variance of f and v the variance of g , the penalty

$$[uX^2 + vY^2]^{0.5} \leq ([u]^{0.5}X + [v]^{0.5}Y)$$

because both sides are positive (variances are positive), and after squaring both sides and subtracting the relationship on the left,

$$0 \leq 2u^{0.5}v^{0.5}XY$$

must be positive, as it consists only of positive elements. A proof of the convexity of the root of the general variance-covariance matrix was given by Kataoka.¹⁴ Figure 2 demonstrates the impact of the linear approximation for the chance constraint given in Figure 1. At the extremes, the approximation is equivalent to the chance constraint. When both decision variables X and Y have values greater than zero, the amount of error is higher for higher levels of probability.

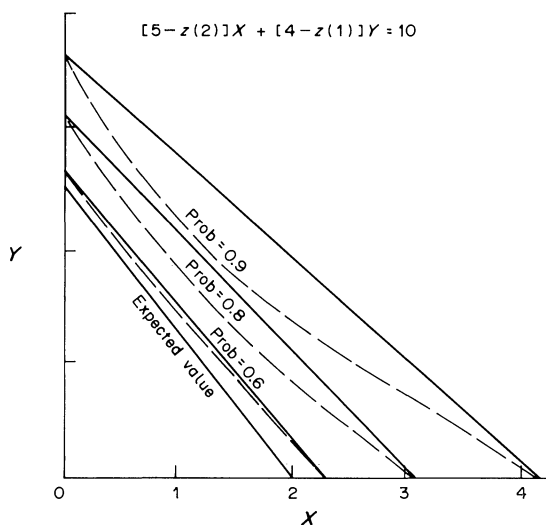


FIG. 2. Linear approximations. - - - True chance-constraints; — linear approximation.

Covariance

While consideration of normally distributed estimates of independent variables has many useful applications, it is sometimes useful to consider the case of chance constraints involving stochastic estimates involving non-independent a_{ij} coefficients. Examples of such applications have been addressed by Freund,¹⁵ Kataoka¹⁴ and others. When covariance exists, the chance constraint is modified by including the covariance terms in the variance-covariance matrix. For the previous function:

$$X + gY + z[uX^2 + 2wXY + vY^2]^{0.5} \leq b,$$

where w is the covariance between F and g .

The linear approximation would still be guaranteed to impose a penalty at least as great as the true chance-constraint. While covariance can taken on negative values, the correlation between covarying variables assures that if w were negative, the linear approximation would still be greater than or equal to the actual chance-constraint.

The definition of the correlation between variables a and b is:

$$\text{Corr}(a, b) = \frac{\text{Covariance}(a, b)}{[\text{Variance}(a)]^{0.5}[\text{Variance}(b)]^{0.5}}.$$

Let $w = \text{covariance}(a, b)$, $u = \text{variance}(a)$, and $v = \text{variance}(b)$. Since $-1 \leq \text{Corr}(a, b) \leq 1$,

$$-1 \leq w/(u^{0.5}v^{0.5}) \leq 1.$$

While w is unrestricted in sign, u and v are strictly positive. Therefore,

$$-u^{0.5}v^{0.5} \leq w \leq u^{0.5}v^{0.5}, \text{ and thus } w^2 < uv.$$

The maximum covariance would be the square root of the product of coefficient variances. The minimum of the covariance would be the negative of that product. Figure 3 presents the impact of the maximum positive and negative covariance for the example chance-constraint at the 0.95 level of confidence. The linear approximation would involve less error if positive covariance existed. The error would be greater in the case of negative covariance. The variance-covariance element $[uX^2 + 2wXY + vY^2] \geq 0$. The worst case is where the covariance is at its maximum negative value. This would be:

$$w = u^{0.5}v^{0.5}.$$

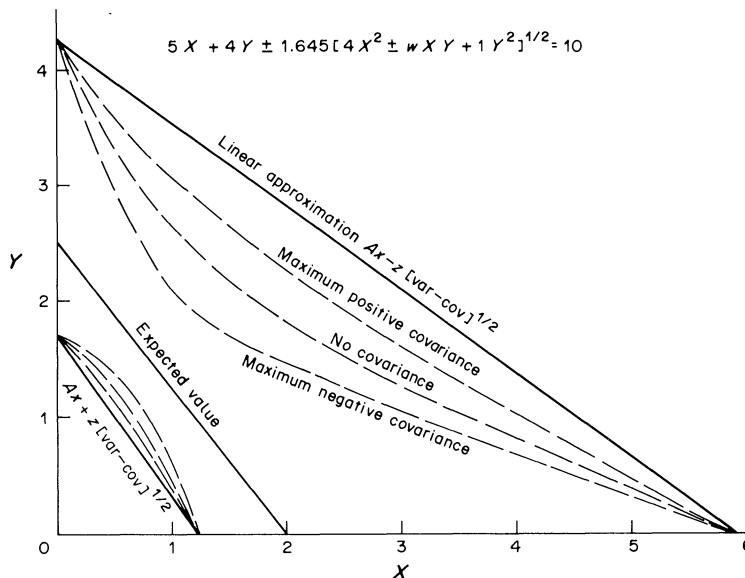


FIG. 3. Impact of covariance.

Let $A = u^{0.5} X$ and let $B = v^{0.5} Y$. The variance-covariance element of the chance constraint would be:

$$A^2 - 2AB + B^2.$$

It does not matter whether A or B is greater, because the order could be rearranged without loss of generality. Let $e = A - B$, supposing that A is the larger of A and B . Therefore, $A = B + e$.

$$(B + e)^2 - 2(B + e)B + B^2$$

$$B^2 + 2Be + e^2 - 2B^2 - 2Be + B^2.$$

Grouping terms yields e^2 as the worst-case value of the variance-covariance element. Since $e^2 \geq 0$, the root is always defined.

DEMONSTRATION OF THE LINEARIZATION

A model presented by Van de Panne and Popp¹² is used to demonstrate the impact of the linearization in a single-objective setting. That model involved the selection of four materials to mix in order to design a cattle-feed mix subject to protein and fat constraints with the objective of minimizing cost. Data was provided for the protein and fat content of each material. Because the protein constraint was probabilistic, variance data was obtained for protein. In this case, the protein content of each material is independent of all other materials selected. Table 1 presents this data.

The deterministic equivalent of this model was:

$$\min \text{cost} = 24.55X_1 + 26.75X_2 + 39.00X_3 + 40.50X_4$$

subject to

$$X_1 + X_2 + X_3 + X_4 = 1$$

$$2.3X_1 + 5.6X_2 + 11.1X_3 + 1.3X_4 \geq 5$$

$$12.0X_1 + 11.9X_2 + 41.8X_3 + 52.1X_4$$

$$-z(0.2809X_1^2 + 0.1936X_2^2 + 20.25X_3^2 + 0.6241X_4^2)^{0.5} \geq 21,$$

where z is the normal functional at prescribed confidence level α . Additional constraints imposing minimum levels on each material are

$$X_j \geq 0.01 \text{ for } j = 1 \text{ to } 4.$$

Results of this model, with varying α levels, are given in Table 2.

TABLE 1. Feed-blend model data

Variable	Protein	Protein variance	Fat content	Cost/ion
X_1 Barley	12.0	0.2809	2.3	24.55
X_2 Oats	11.9	0.1936	5.6	26.75
X_3 Sesame flakes	41.8	20.2500	11.1	39.00
X_4 Groundnut meal	52.1	0.6241	1.3	40.50

Source: Van de Panne and Popp,¹² p. 406.

TABLE 2. Chance-constrained model results

Target probability	X_1	X_2	X_3	X_4	Cost	True probability
0.5 (LP)	0.6500	0.0513	0.2887	0.0100	28.99	0.50
0.6	0.6696	0.0199	0.3005	0.0100	29.10	0.60
0.7	0.6687	0.0100	0.3049	0.0164	29.24	0.70
0.8	0.6571	0.0100	0.3061	0.0268	29.42	0.80
0.85	0.6499	0.0100	0.3068	0.0333	29.54	0.85
0.9	0.6409	0.0100	0.3078	0.0414	29.68	0.90
0.95	0.6274	0.0100	0.3091	0.0534	29.89	0.95
0.98	0.5874	0.0401	0.3000	0.0725	30.13	0.9798
0.99	0.1791	0.5291	0.1271	0.1647	30.18	0.9819

Note: numbers rounded.

Linear approximation development

The linear approximation at any α level (>0.5) can be used to replace the non-linear chance constraints of the type used in the example. For a 0.8 level of confidence:

$$\text{Variance} \leq [0.2809X_1^2 + 0.1936X_2^2 + 20.25X_3^2 + 0.6241X_4^2]^{0.5}.$$

From above, we know that

$$(0.2809X_1^2)^{0.5} + (0.1936X_2^2)^{0.5} + (20.25X_3^2)^{0.5} + (0.6241X_4^2)^{0.5} \leq [0.2809X_1^2 + 0.1936X_2^2 + 20.25X_3^2 + 0.6242X_4^2]^{0.5}.$$

The chance constraint is the fourth constraint of the model. The normal density functional for 0.8 is 0.841. Therefore, the a_{ij} coefficients for the approximation of the chance constraint are:

$$\begin{aligned} a_{i1} &= \{12.0 - 0.841(0.2809^{0.5})\} = 11.55427 \\ a_{i2} &= \{11.9 - 0.841(0.1936^{0.5})\} = 11.52996 \\ a_{i3} &= \{41.8 - 0.841(20.25^{0.5})\} = 38.0155 \\ a_{i4} &= \{52.1 - 0.841(0.6241^{0.5})\} = 51.43561. \end{aligned}$$

Trade-off development

Chance constraints in their true form provide a penalty on the attainment of other objectives. The linear approximation provides a penalty at least as great as the chance constraints. The results at the given probability levels are given in Table 3.

The attained probability can be seen increasingly to exceed the target as the target probability approaches 1. At a probability level of 0.6, the penalty function resulted in a solution for the chance constraint with a true probability of 0.622. Calculating error as the proportion of available probability, this means the target probability was overachieved by $0.022/(1 - 0.6)$, or 5.5%. At a target of 0.99, however, the attained probability was 0.999, or 90% of the possible error. The relative error as a function of target probability is portrayed in Figure 4.

TABLE 3. Approximation results by probability level

Target probability	X_1	X_2	X_3	X_4	Cost	True probability
0.6	0.6744	0.0123	0.3033	0.0100	29.12	0.622
0.7	0.6643	0.0100	0.3054	0.0204	29.31	0.741
0.8	0.6499	0.0100	0.3069	0.0333	29.54	0.850
0.85	0.6409	0.0100	0.3078	0.0413	29.68	0.900
0.9	0.6296	0.0100	0.3089	0.0515	29.86	0.944
0.95	0.6127	0.0100	0.3106	0.0666	30.12	0.979
0.98	0.5935	0.0100	0.3126	0.0839	30.43	0.995
0.99	0.0100	0.6995	0.0696	0.2209	30.62	0.999

Note: numbers rounded.

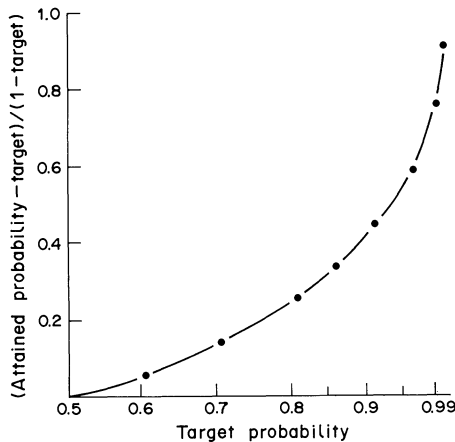


FIG. 4. Relative error of approximation by target probability.

CONCLUSION

The linear approximation to chance constraints can be a useful tool for chance-constrained models. The formulation proposed here obtains approximate solutions to continuous chance-constrained models that will achieve at least the prescribed confidence level. A number of recent studies have applied chance-constrained models, using added information of coefficient estimates, to include variance. These studies required special algorithms to solve the resulting non-linear constraint set. The approximation presented would allow approximate solution to such models with readily available linear codes. The cost of this technique is inaccuracy. The resulting error of that approximation has been discussed.

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