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The impact of distribution on value-at-risk measures

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ABSTRACT

Value at risk is a popular approach to aid financial risk management. Questions about the appropriateness of the measure have arisen since the related 2008 bubble collapses in some US housing markets and the global financial market. These questions include the presence of fat tails and their impact. This paper compares results based upon assumptions of normality and logistic distributions, comparing portfolios generated with various probabilistic models. Computations are applied to real stock data. Optimization models are described, with simulation models evaluating comparative model performance. Chi-square tests indicated that logistic distribution better fit the data than the normal distribution. The error implied by value-at-risk assumptions is demonstrated through Monte Carlo simulation.

1. Introduction

Risk management has provided many tools to evaluate the chance of loss. Hubbard [1] defines risk management as the identification, assessment, and prioritization of risks followed by coordinated and economical application of resources to minimize, monitor, and control the probability and/or impact of unfortunate events. Taleb [2] condenses this to being prepared for all relevant eventualities. This is a comprehensive view of risk management, covering all possible risks facing an organization. The fact is that one cannot expect compensation or profit without taking on some risk. The key to successful risk management is to select those risks that one is competent to deal with, and to find some way to avoid, reduce, or insure against those risks not in this category.

Markowitz [3] equated risk with variance, which would be controlled by diversification, considering correlation across the investments that are available. His models focused on identifying efficient portfolios non-dominated with respect to risk and return. This leads to the need for some calculus of preferences, such as multi-attribute utility theory ([4] as one source among hundreds). Financial risk management has developed additional tools such as value at risk.

Value at risk (VaR) is one of the most widely used models in risk management [5]. It is based on probability and statistics [6]. VaR can be characterized as a maximum expected loss (a point estimate), given some time horizon and within a given confidence interval. Its utility is in providing a measure of risk that illustrates the risk inherent in a portfolio with multiple risk factors, such as portfolios held by large banks, which are diversified across many risk factors and product types. VaR is used to estimate the boundaries of risk for a portfolio over a given time period, for an assumed probability distribution of market performance. It is a point estimate based upon the assumed probability distribution. The purpose is to diagnose risk exposure.

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Definition. VaR describes the probability distribution for the value (earnings or losses) of an investment (firm, portfolio, etc.). The mean is a point estimate of a statistic, showing historical central tendency. VaR is also a point estimate, but offset from the mean. It requires specification of a given probability level, and then provides the point estimate of the return or better expected to occur at the prescribed probability.

However, VaR has undesirable properties, especially for gain and loss data with non-elliptical distributions. It satisfies the well-accepted principle of diversification under normal distribution. However, it violates the fairly accepted sub-additive rule; i.e., the portfolio VaR is not smaller than the sum of component VaR. The reason is that VaR only considers the extreme percentile of a gain/loss distribution without considering the magnitude of the loss. As a consequence, a variant of VaR, usually labeled *Conditional-Value-at-Risk* (or CVaR), has been used. In computational issues, optimization of CVaR can be very simple, which is another reason for the adoption of CVaR. This pioneer work was initiated by Rockafellar and Uryassev [7], where CVaR constraints in optimization problems can be formulated as linear constraints. CVaR represents a weighted average between the VaR and losses exceeding the VaR. CVaR is a risk assessment approach used to reduce the probability a portfolio will incur large losses assuming a specified confidence level. It is possible to maximize portfolio return subject to constraints including CVaR and other downside risk measures, both absolute and relative to a benchmark (market and liability based). Simulation based models to optimize CVaR under controlled conditions can be developed.

2. Model development

Our study examines various formulations from chance constrained programming [8] with respect to risk-minimization decisions in investment. Optimization models are generated with Excel Solver (which uses the generalized reduced gradient algorithm for nonlinear optimization). We compare three traditional models of VaR and CVaR based on assumed distributions. We verify model results with Monte Carlo simulation using Crystal Ball software.

2.1. Monte Carlo simulation of VaR and CVaR

Simulation models are sets of assumptions concerning the relationship among model components. Among these assumptions are probability distributions, such as a normal distribution (with parameter for a mean), lognormal (parameters mean and variance), or any of a number of other distributions. A simulation run is a sample from an infinite population of possible results for a given model. After a simulation model is built, a selected number of trials can be established. Statistical methods can be used to validate simulation models and design simulation experiments.

Many financial simulation models can be accomplished on spreadsheets, such as Excel. There are a number of commercial add-on products that can be added to Excel, such as @Risk, Crystal Ball, or Frontline Solver that vastly extend the simulation power of spreadsheet models [9]. These add-ons make it very easy to replicate simulation runs, and include the ability to correlate variables, expeditiously select from standard distributions, aggregate and display output, and other useful functions.

2.2. Portfolio optimization models

This section discusses traditional portfolio optimization models based on mean–variance framework and chance constraints. As before, denote by y_i the annual return rate of investing in Security *i*. Denote by σ_i^2 the variance of Security *i* and σ_{ij} the variance–covariance between Security *i* and j. The first approach is to maximize the expected return, a linear programming model with a trivial solution (invest everything in the investment alternative with the highest expected return). This option of course usually involves high risk. For purposes of modeling, we use daily average returns for the investment options used. As the daily returns are quite small, we use an investment pool of 1000 currency units:

$$\text{Max Expected Return} = \sum_{i=1}^{n} y_i w_i$$

$$\text{Subject to :} \quad \sum_{i=1}^{n} w_i \leq 1000$$

$$w_i \geq 0 \quad \text{for all } j$$

where w_i is the amount invested in investment option *i* and y_i is the return from investment option *i*.

The second model is to minimize variance:

$$\text{Min Variance} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

$$\text{Subject to :} \quad \sum_{i=1}^{n} w_i \leq 1000$$

$$w_i \geq 0 \quad \text{for all } i.$$

The third optimization model we consider is to maximize the value at the 0.95 level of attainment, which is equivalent to maximizing the target in the third model, with a specified α (we used 0.95):

$$\text{Max Target}$$

$$\text{Subject to}: \quad \sum_{i=1}^{n} w_i \leq 1000$$

$$V = \sum_{i=1}^{n} y_i w_i - t_{0.05,\text{d.f.}} \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \left(w_i w_j \sigma_{ij}\right)}$$

$$w_i \geq 0 \quad \text{for all } i.$$

This would be equivalent to a model maximizing VaR if VaR were defined in terms of the median when assuming t-distribution in return data. If we define conditional value at risk, CVaR, as the median of the outcomes worse than VaR, we could utilize a $t_{0.025,\text{d.f.}}$ in the penalty function, which would be equivalent to maximizing the limit at the 0.975 probability level.

Similarly, we can develop models (2) maximizing the expected portfolio value subject to both CVaR and chance constraints, and (3) maximizing probability of satisfying a chance constraint set.

All the three models include a common general chance constraint set, allowing probabilistic attainment of functional levels:

$$P\{-w^T y_j \leq -R_j\} \geq \beta_j.$$

This set is nonlinear, requiring a nonlinear programming solution. This inhibits the size of the model to be analyzed, as large values of model parameters m (number of constraints) and especially n (number of variables) make it much harder to obtain a solution.

The fourth type of model is a chance constrained model:

$$\text{Max Expected Return} = \sum_{i=1}^{n} y_i w_i$$

$$\text{Subject to}: \quad \sum_{i=1}^{n} w_i \leq 1000$$

$$\Pr\left\{\sum_{i=1}^{n} y_i w_i = \text{target}\right\} = \alpha$$

$$w_i \geq 0 \quad \text{for all } i.$$

The chance constraint $\Pr\left\{\sum_{i=1}^{n} y_i w_i = \text{target}\right\} = \alpha$ is equivalent to:

$$\sum_{i=1}^{n} y_i w_i - t_{\alpha,\text{d.f.}} \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \left(w_i w_j \sigma_{ij}\right)} = \text{target}.$$

We used targets of 1000 (investment breakeven), 950, and 900.

3. Generating solutions

Data was gathered from Web sources over five investment options: Morgan Stanley World Index (MSCI), New York Stock Exchange Composite Index (NYSE), Standard & Poors 500 (S&P), Shenzhen Composite (China), and Eurostoxx50 (Euro). Daily data for a period was split into two equal groups of 370 observations each. The data set labeled "pre-bubble" was from 6/29/2006 through 1/2/2007, while the data set labeled "post-bubble"covered 1/3/2007 through 7/6/2009. Data shown in Table 1 is for daily change multiplied by 100, as using daily change led to too many decimal places to detect results from the software. Models were developed in Excel, using annual data for means, variances, and covariances. The data is given in Table 1.

Clearly, the post-bubble period was much worse, with an average loss for each investment option, and a much larger standard deviation.

3.1. Distribution fitting

Data was tested for distribution using Crystal Ball, which can test fourteen continuous distributions. The data exhibited fat tails. This was expected, as Hubbard [1] and many others have pointed out. The error caused is that if the normal distribution is assumed, and the distribution spreads out more (thus fat tails), decision makers falsely assume less risk than actually is

Table 1
Data measures—annual return from investment (100 times daily).

Pre-bubble	MSCI	NYSE	S&P	China	Eurostoxx
Mean	0.063	0.058	0.045	0.359	0.062
St Dev	0.749	0.953	0.900	2.092	0.963
Min	−2.484	−3.630	−3.472	−8.543	−2.886
Max	2.025	3.136	2.921	8.709	2.929
Post-bubble					
Mean	−0.121	−0.105	−0.098	−0.064	−0.138
St Dev	2.044	2.613	2.488	2.755	2.402
Min	−7.063	−9.756	−9.035	−11.924	−7.880
Max	9.523	12.216	11.580	8.888	11.002

Table 2
Relative distribution fits (probabilities from chi-square tests).

Pre-bubble	MSCI	NYSE	S&P	China	Eurostoxx
Logistic	**0.356**	**0.023**	**0.001**	0.003	**0.475**
Student-*t*	0.036	0.005	0.000	**0.009**	0.053
Normal	0.006	0.000	0.000	0.000	0.021
Lognormal	0.004	0.000	0.000	0.000	0.014
Post-bubble					
Logistic	**0.034**	**0.050**	**0.057**	**0.266**	**0.063**
Student-*t*	0.007	0.003	0.114	0.046	0.023
Normal	0.000	0.000	0.000	0.017	0.000
Lognormal	0.000	0.000	0.000	0.012	0.000

Table 3
Distribution fit parameters (100 times daily).

Pre-bubble	MSCI	NYSE	S&P	China	Eurostoxx
Logistic (mean, scale)	0.09, 0.41	0.09, 0.50	0.07, 0.47	0.49, 1.10	0.07, 0.53
Student-*t* (mean, stdev, d.f.)	0.06, 0.70 14.5	0.06, 0.82, 7.5	0.04, 0.76 6.9	0.36, 1.72, 6.2	0.06, 0.91, 17.6
Normal (mean, stdev)	0.06, 0.75	0.06, 0.95	0.04, 0.90	0.36, 2.09	0.06, 0.96
Lognormal (location, mean, stdev)	−5510, 0.06, 0.75	−8642, 0.06, 0.95	−8075, 0.04, 0.90	−25, 0.36, 2.21	−8047, 0.06, 0.96
Post-bubble					
Logistic (mean, scale)	−0.09, 1.05	−0.09, 1.35	−0.10, 1.28	0.03, 1.50	−0.15, 1.24
Student-*t* (mean, stdev, d.f.)	−0.12, 1.67, 6.0	−0.10, 2.14, 6.0	−0.10, 2.02, 5.9	−0.06, 2.43, 8.9	−0.14, 1.96, 5.99
Normal (mean, stdev)	−0.12, 2.04	−0.10, 2.61	−0.10, 2.49	−0.06, 2.75	−0.14, 2.40
Lognormal (location, mean, stdev)	−27 065, −0.12, 2.04	−182, −0.10, 2.61	−111, −0.10, 2.49	−25 134, 0.06, 2.75	−55, −0.14, 2.40

present. In testing the data for distributions, we checked each of the five investments (by pre-bubble and post-bubble sets). We used the Crystal Ball option to check all the fourteen continuous distributions. Relative fits of the distributions that had the best chi-square scores are shown in Table 2.

The logistic distribution proved the best fit in both subsets of data, each containing 370 observations. There was one case where the student-*t* distribution had a slightly better fit than the logistic distribution (pre-bubble China) using Chi-square (Kolmogorov–Smirnov and Anderson–Darling tests had better fit for logistic). The fits were usually better for the pre-bubble data subset than for the post-bubble data subset, although none were extremely strong. Normal and lognormal distributions had similar fits, but the lognormal distribution had fewer degrees of freedom and thus had a slightly lower probability of fit for the Chi-square test than did the normal distribution. Table 3 gives distribution parameters for each data series.

Our interest now is on the impact of assuming the normal distribution, using the pre-bubble data to model the data, testing it on the post-bubble data. Fig. 1 shows the logistic distribution fit for MSCI over the pre-bubble period, Fig. 2 for student-*t*, and Fig. 3 for normal.

The fits look similar, and the 0.95 VaR for each is quite similar. However, the minimum statistics for each are −5.19 for logistic, −3.43 for student-*t*, and −2.67 for normal, demonstrating thicker tails for the logistic and student-*t* distributions than for the normal distribution. Table 4 compares statistics for these distributions generated through simulation.

The logistic distribution had a wider spread (due to the presence of fatter tails), confirmed by the kurtosis measures.

3.2. Generation of portfolios

Models were generated and solved with Excel Solver with the following objective functions.

(1) Maximize expected return s.t. budget ≤ 1.
(2) Minimize variance s.t. investment $= 1$.
(3) Maximize VaR for $\alpha = 0.99, 0.95$, and 0.9.

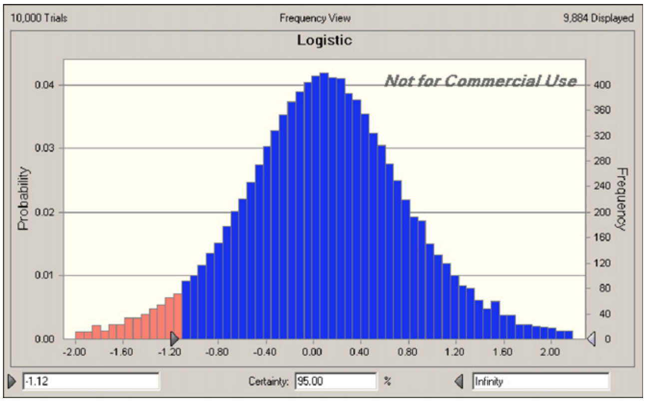

Fig. 1. Logistic fit for MSCI—pre-bubble.

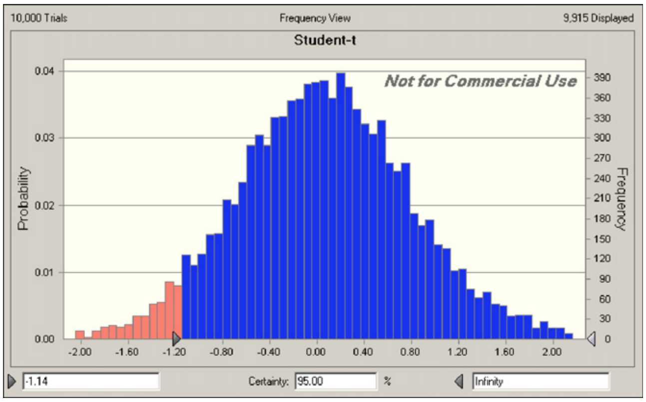

Fig. 2. Student-*t* fit for MSCI—pre-bubble.

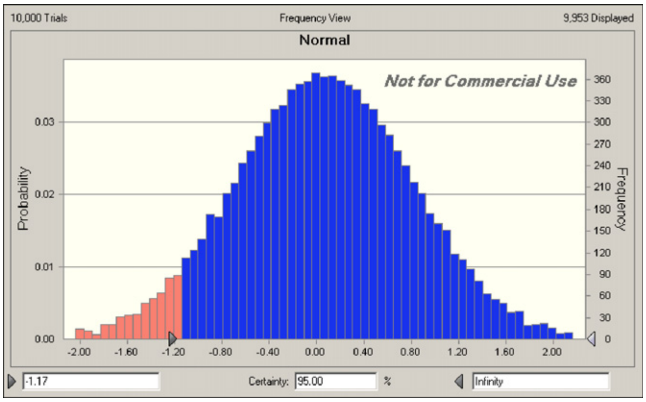

Fig. 3. Normal fit for MSCI—pre-bubble.

Table 4
Statistics for MSCI pre-bubble distributions generated— daily return.

Statistic	Logistic (0.09, 0.41)	Student-*t* (0.06, 0.70, 14.5)	Normal(0.06, 0.75)
Mean	0.09	0.06	0.06
Standard deviation	0.74	0.08	0.75
Skewness	−0.0278	−0.0195	0.0019
Kurtosis	4.31	3.49	3.00
Minimum	−5.19	−3.43	−2.70
Maximum	3.88	3.10	3.11

Table 5
Model results based on logistic assumption.

Objective	*MSCI*	*NYSE*	*S&P*	*China*	*Euro*
Max *E*[return]	0	0	0	1	0
Min Variance	0.884	0	0.029	0.087	0
Min VaR(0.99)	0.882	0	0.013	0.105	0
Min VaR(0.95)	0.881	0	0.004	0.115	0
Min VaR(0.9)	0.875	0	0	0.125	0
Max Pr{*E*[Ret] > 1}	0.529	0	0	0.471	0
Max Pr{*E*[Ret] > 0.95}	0.668	0	0	0.332	0
Max Pr{*E*[Ret] > 0.90}	0.734	0	0	0.266	0
CC {Pr > 0.6[Ret > 1]}	0.047	0	0	0.946	0.007

Table 6
Simulated post-bubble model annual return—logistic.

Objective	Return	Variance	VaR(0.99)	VaR(0.95)	VaR(0.90)	Pr{>1}	Pr{>0.95}	Pr{>0.9}
Max *E*[return]	**0.359**	4.364	4.933	3.032	2.172	0.5773	0.5878	0.5983
Min Variance	0.088	**0.526**	1.749	1.089	0.791	0.5547	0.5853	0.6153
Min VaR(0.99)	0.093	0.527	**1.747**	1.086	0.786	0.5581	0.5886	0.6185
Min VaR(0.95)	0.097	0.530	1.747	**1.085**	0.785	0.5599	0.5904	0.6201
Min VaR(0.9)	0.100	0.533	1.750	1.085	**0.785**	0.5616	0.5919	0.6215
Max Pr{*E*[Ret] > 1}	0.202	1.205	2.579	1.580	1.128	**0.5827**	0.6026	0.6222
Max Pr{*E*[Ret] > 0.95}	0.161	0.803	2.109	1.294	0.924	0.5807	**0.6051**	0.6290
Max Pr{*E*[Ret] > 0.90}	0.142	0.675	1.940	1.192	0.854	0.5775	0.6041	**0.6302**
CC {Pr > 0.6[Ret > 1]}	0.343	3.921	4.674	2.872	2.056	0.5778	0.5890	**0.6000**

Table 7
Model results based on normal assumption.

Objective	*MSCI*	*NYSE*	*S&P*	*China*	*Euro*
Min VaR(0.99)	0.882	0	0.0115	0.1065	0
Min VaR(0.95)	0.881	0	0.004	0.115	0
Min VaR(0.9)	0.877	0	0	0.123	0
Max Pr{*E*[Ret] > 1}	0	0	0.174	0.423	0.403
Max Pr{*E*[Ret] > 0.95}	0.517	0.001	0	0.343	0.139
Max Pr{*E*[Ret] > 0.90}	0.734	0	0	0.266	0
CC {Pr > 0.6[Ret > 1]}	0.695	0	0	0.305	0

(4) Maximize probability{return > specified level} for levels [1, 0.95, and 0. 90].
(5) Maximize expected return s.t. probability{return = specified level} = α for return of 1 and α [0.6].

Excel Solver is capable of nonlinear optimization, using generalized reduced gradient methods. Table 5 gives the solutions obtained.

Looking at the solutions, the Chinese stock index clearly had the greatest return and the greatest risk. The New York Stock Exchange index was never selected in this set of runs. The Morgan Stanley index was often selected as a means to lower various risk measures.

4. Monte Carlo simulation

Crystal Ball simulation software was used to evaluate results. The difference between the optimization model inputs and the inputs for Monte Carlo simulation was that the Monte Carlo simulation modeled a full year of daily changes for each portfolio (245 trading days). This was expected to provide greater accuracy. Further, logistic distributions were assumed in the Monte Carlo runs, as Crystal Ball says to use normal for $n > 30$, but we have a much larger n, and we want the distribution to have fat tails. The results for the simulations are given in Table 6.

In Table 6, optimal solutions are shown in bold. Simulation results were consistent with the theoretical outcomes. As should be the case, the diagonal is the optimal solution for the first nine models. Clearly maximizing return came with a high degree of risk, as indicated in the solution's high variance. This also resulted in the highest value-at-risk calculations, and the lowest probability of not losing money. The greatest probability of retaining the original investment was generated by the solution designed for that very purpose. In the case of the chance constrained model, the bold figure demonstrates that the chance constraint is binding, yielding the greatest return subject to a 0.6 probability of retaining at least 90% of the initial investment.

Had the normal distribution been used to generate portfolios, the solutions maximizing return and minimizing variance would have been the same as given in Table 5. The other solutions are given in Table 7.

The results from simulation (assuming logistic outcome in the post-bubble period) are shown in Table 8.

Table 8
Simulated post-bubble model annual return—normal.

Objective	Return	Variance	VaR(0.99)	VaR(0.95)	VaR(0.90)	Pr{>1}	Pr{>0.95}	Pr{>0.9}
Min VaR(0.99)	**0.094**	0.528	1.747	1.179	0.880	0.5584	0.5889	0.6188
Min VaR(0.95)	0.097	**0.530**	1.747	1.182	0.882	0.5597	0.5903	0.6201
Min VaR(0.9)	0.099	0.532	**1.749**	1.085	0.785	0.5540	0.5809	0.6075
Max Pr{E[Ret] > 1}	0.185	1.093	2.464	**1.513**	1.082	0.5794	0.6004	0.6210
Max Pr{E[Ret] > 0.95}	0.164	0.837	2.154	1.321	**0.944**	0.5806	0.6045	0.6280
Max Pr{E[Ret] > 0.90}	0.142	0.675	1.940	1.192	0.854	**0.5775**	0.6041	0.6302
CC {Pr > 0.6[Ret > 1]}	0.153	0.746	2.035	1.249	0.893	0.5703	**0.5929**	0.6152

The point we are emphasizing is that the solutions generated assuming the logistic distribution would have yielded less loss as measured by the value-at-risk. Using the VaR level of 0.99, there is practically no difference. But at the 0.90 level, the amount that would have been lost at the 90th percentile level would have been worse 10% of the time. Thus we assume that the logistic distribution would have saved investor's money in the period of dismal return.

5. Conclusion

The primary outcome of this research confirms widely expressed opinions that financial return data has fat tails, and is better fit by the logistic distribution than the normal distribution. The impact is that the risk is understated when assuming the normal distribution.

VaR is a useful concept in terms of assessing probabilities of investment alternatives. It is a point estimator, like the mean (which could be viewed as the VaR for a probability of 0.5). It is only as valid as the assumptions made, which include the distributions used in the model and the parameter estimates. However, VaR and CVaR provide useful tools for financial investment. Monte Carlo simulation provides a flexible mechanism to measure value at risk for any given assumption.

This paper focuses on the tradeoffs between two approaches to optimize return subject to constraints on risk: conditional value-at-risk and chance constraints. We controlled for random numbers, but simulating such a complex model makes replication problematic. We utilized Crystal Ball to establish the best fit distribution, finding logistic best in most cases. This matched our expectations, in that it includes the ability to fatten tails. The student-t distribution had the second best fit in most cases, and had the best fit for the Chinese index in the pre-bubble data, while normal and lognormal were either third or fourth in all cases. We did not check the power-law distribution, which Hubbard [1] argues is better, because Crystal Ball does not have that distribution in its palette of distributions. One of the most difficult problems was including correlation, which is very important in investments of this type. While Crystal Ball has the ability to correlate, it is difficult to correlate across 245 days over 5 variables. (Real problems would of course have many more investment alternatives available.) We checked the fit provided, and found it to be reasonable.

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