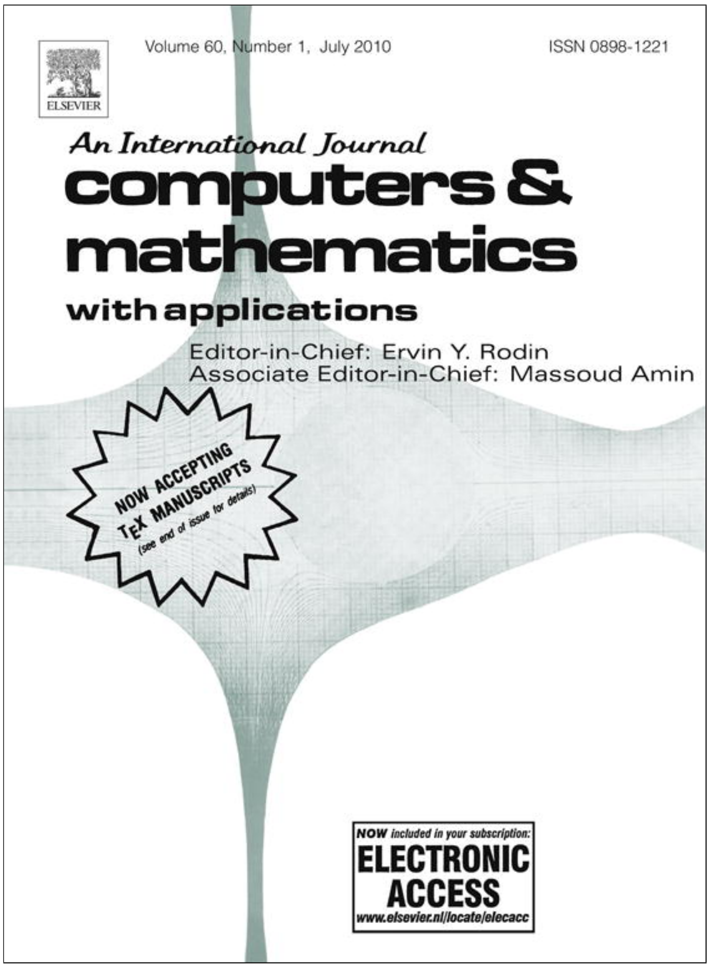
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Fuzzy multiattribute grey related analysis using DEA

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ABSTRACT

Qualitative input is encountered in many real decision-making domains. Often such domains include tradeoffs among multiple attributes, and estimates of parameters are often expressed with some degree of uncertainty. Grey related analysis has been proposed as a means to use interval fuzzy representation of data. When dealing with multiple criteria data, model parameters that can involve uncertainty include both performances of alternatives on attributes as well as attribute weights. DEA is proposed as an objective way to derive weights. This paper presents a grey-related fuzzy set methodology incorporating data envelopment analysis as a way to more objectively rank alternatives. The purpose is to demonstrate the method. The focus is on identifying alternatives performing the most efficiently with respect to the decision maker's preference. The method is demonstrated on a multiattribute siting problem. Simulation is applied to validate model efficiency.

1. Introduction

Uncertainty and the presence of multiple, often conflicting objectives are common in real organizational decision-making. Fuzzy sets [1] have been very useful in representing the uncertain performance of different systems measured over different attributes. Fuzzy sets provide a realistic and practical way to describe many aspects of the real world. Interval-valued fuzzy (or vague) sets [2] are essentially identical to the concept of intuitionist fuzzy sets [3]. Interval-valued fuzzy sets in recent years have been applied to many fields, to include fuzzy controlling [4], distributed decision-making [5], and other fields. Interval-valued fuzzy sets have been shown to provide improved performance over other fuzzy set approaches in some contexts [6]. There have been a number of extensions to the concept of fuzzy sets, including vague (or interval-valued) fuzzy sets and grey-related fuzzy set theory [7].

DEA was first introduced in 1978 by Charnes, Cooper and Rhodes [8] for efficiency analysis of Decision-making Units (DMU). DEA can be used for modeling operational processes, and its empirical orientation and absence of *a priori* assumptions have resulted in its use in a number of studies involving efficient frontier estimation in both nonprofit and in private sectors. DEA has become a leading approach for efficiency analysis in many fields [9], such as education, health care, banking, armed forces, retail outlets, transportation, manufacturing and so forth.

Fuzzy sets and DEA have been combined in a number of recent publications. Karsak [10] combined the use of DEA to identify efficient solutions with the subsequent use of fuzzy sets to rank the identified efficient alternatives. DEA has been used by multiple researchers to identify efficient alternatives that were described by fuzzy scores [11,12]. DEA has also been

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combined with fuzzy scoring to identify both optimistic and pessimistic sets of efficient solutions [13,14]. Fuzzy multiple objective linear programming has been used to implement DEA over imprecise (fuzzy) data [15]. Omero et al. [16] recently integrated DEA with qualitative and expert assessment by applying fuzzy logic.

We present the Method of grey related analysis as a means to reflect uncertainty in multi-attribute models through interval numbers. The grey system theory was developed by Deng [17], based upon the concept that information is sometimes incomplete or unknown. In recent work, classical grey related analysis has been used to handle fuzzy MADM problems by introducing interval data as input [18]. Simulation can more completely describe possible outcomes of fuzzy models [19]. However, allowing fuzzy data directly may be inappropriate in grey related analysis due to the following reason. Classical grey related analysis uses average connection coefficients. This means that the importance of different connections cannot be identified as we assign equal weights to each connection. In this case, it is attractive to derive an objective weight for each connection. As a data-oriented approach, data envelopment analysis (DEA [20–23]) provides an alternative solution to this problem.

This paper presents a grey-related fuzzy set methodology incorporating data envelopment analysis as a way to more objectively rank alternatives. Section 2 reviews DEA pertinent to this paper. Section 3 presents the grey related method. Section 4 demonstrates the method on a hypothetical waste dump site selection problem, which reflects multiple and conflicting criteria. Section 5 analyzes results through a simulation model, and conclusions are presented in Section 6.

2. DEA and multiple attribute decision-making models

This section briefly investigates literature about DEA and multiple criteria decision-making (MCDM) models. The connection of the MCDM problem and DEA was first discussed by Golany [24], where he combined interactive, multiple-objective linear programming and DEA. DEA has many similarities with many standard multi-criteria methods like the Analytic Hierarchy Process (AHP), PROMETHEE, the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), Compromise Programming (CP) and Simple Product Weighting (SPW) [25]. The first similarity concerns weights. A weighting scheme is usually required by TOPSIS, SPW and AHP. Weight constraints can also be imposed in DEA as a representation of the decision maker's preference information over different criteria. The second similarity can be in the ranking of alternatives. MCDM, like TOPSIS, SPW and AHP, provides a rank order of alternatives. While ranking alternatives is not the ultimate aim of DEA, some DEA models such as super-efficiency DEA do provide a good approach for rating the alternatives. DEA can also be appropriate for handling fuzzy multiattribute problems [26]. To accomplish this, DEA has been used with standard multi-criteria methods [27].

Out of all the literature addressing MCDM and DEA problems, very few articles discuss fuzzy multiattribute problems with DEA. Nothing in existing literature discusses grey related analysis and DEA. To investigate the performance of DEA and grey related analysis in the fuzzy multiattribute problem domain will fill this research gap.

3. Grey related analysis with data envelopement analysis

Grey related analysis is can be applied to both fuzzy and crisp data. Here we use it as a means to obtain a solution from fuzzy data. A general formulation for a multiattribute model is:

$$\text{value}_j = \sum_i w_i \times u\left(x_{ij}\right) \tag{1}$$

where each alternative j is measured in value terms $u(x_{ij})$ over i attributes given relative weights of each attribute w_i. These relative weights may express relative importance, and if scores are not standardized, also reflect relative scale.

In fuzzy domains, both w_i and $u(x_{ij})$ may involve uncertainty. If a multiple attribute decision-making problem with interval numbers has m feasible plans $X_1, X_2, \ldots, X_m$ and n indexes $G_1, G_2, \ldots, G_n$ and the index value of j-th index G_j of alternative X_i is an interval number $[a_{ij}^-, a_{ij}^+]$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Weights can also be expressed as interval numbers where w_i is in $\left[w_i^-, w_i^+\right]$. The multiple attribute decision-making problem with interval numbers is called a multiple attribute decision-making problem with interval-valued indexes.

To overview the steps in our approach, Steps 1 and 2 prepare the data, and Step 3 eliminates scale differences. Step 4 provides an ideal vector. Step 5 calculates connection coefficients based on decision maker selection of the distinguishing coefficient. Step 6 generates objective weights through an optimization model, the results of which are used in Step 7 to rank alternatives.

Generally, the grey related analysis Method with DEA uses several steps as follows:

Step 1: *Construct decision matrix A with index number of interval numbers.*

If the index value of j-th index G_j of feasible plan X_i is an interval number $[a_{ij}^-, a_{ij}^+]$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, the decision matrix A with the index number of interval numbers is defined as the following:

$$A = \begin{bmatrix} [a_{11}^-, a_{11}^+] & [a_{12}^-, a_{12}^+] & \cdots & [a_{1n}^-, a_{1n}^+] \\ [a_{21}^-, a_{21}^+] & [a_{22}^-, a_{22}^+] & \cdots & [a_{2n}^-, a_{2n}^+] \\ \cdots & \cdots & \cdots & \cdots \\ [a_{m1}^-, a_{m1}^+] & [a_{m2}^-, a_{m2}^+] & \cdots & [a_{mn}^-, a_{mn}^+] \end{bmatrix}. \tag{2}$$

Step 2: *Transform all "contrary indexes" into positive indexes.*

The index is called a positive index if a greater index value is better. The index is called a contrary index if a smaller index value is better. We may transform contrary indexes into positive indexes if j-th index G_j is a contrary index

$$[b_{ij}^-, b_{ij}^+] = [-a_{ij}^+, -a_{ij}^-], \quad i = 1, 2, \ldots, m. \tag{3}$$

Without loss of generality, in the following we supposed that all the indexes are "positive indexes".

Step 3: *Standardize decision matrix A with index number of interval numbers to gain standardizing decision matrix* $R = [r_{ij}^-, r_{ij}^+]$.

If we mark the column vectors of decision matrix A with interval-valued indexes with $A_1, A_2, \ldots, A_n$, the element of standardizing decision matrix $R = [r_{ij}^-, r_{ij}^+]$ is defined as follows:

$$[r_{ij}^-, r_{ij}^+] = \frac{[a_{ij}^-, a_{ij}^+]}{\|A_j\|}, \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n. \tag{4}$$

Here $\|A_j\|$ is the possible range of values for each column of the decision matrix. Without loss of generality, we can use the maximum entry for each column of A if 0 would be the lowest possible score a_{ij}.

Step 4: *Determine reference number sequence.*

The element of reference number sequence is composed of the optimal weighted interval number index value of every plan.

$U_0 = ([u_0^-(1), u_0^+(1)], [u_0^-(2), u_0^+(2)], \ldots, [u_0^-(n), u_0^+(n)])$ is called a reference number sequence if $u_0^-(j) = \max_{1\le i\le m} r_{ij}^-$, $u_0^+(j) = \max_{1\le i\le m} r_{ij}^+$, $j = 1, 2, \ldots, n$.

Step 5: *Calculate the connection between the sequence composed of interval number standardizing index value of every plan and reference number sequence.*

First, calculate the connection coefficient $\xi_i(k)$ between the sequence composed of interval number standardizing index value of every plan $U_i = ([c_{i1}^-, c_{i1}^+], [c_{i2}^-, c_{i2}^+], \ldots, [c_{in}^-, c_{in}^+])$ and reference number sequence $U_0 = ([u_0^-(1), u_0^+(1)], [u_0^-(2), u_0^+(2)], \ldots, [u_0^-(n), u_0^+(n)])$. The formula of $\xi_i(k)$ is:

$$\xi_i(k) = \frac{\min\limits_i \min\limits_k |[u_0^-(k), u_0^+(k)] - [c_{ik}^-, c_{ik}^+]| + \rho \max\limits_i \max\limits_k |[u_0^-(k), u_0^+(k)] - [c_{ik}^-, c_{ik}^+]|}{|[u_0^-(k), u_0^+(k)] - [c_{ik}^-, c_{ik}^+]| + \rho \max\limits_i \max\limits_k |[u_0^-(k), u_0^+(k)] - [c_{ik}^-, c_{ik}^+]|}. \tag{5}$$

Here, $\rho \in [0, 1]$ is called a distinguishing coefficient. The smaller ρ is, the greater its distinguishing power. In general, the value of ρ may change according to the practical situation. The classical grey-related parameter, ρ is equivalent to the proportion of emphasis given to the max function. The sensitivity of such a parameter to the final results has been discussed in existing work [19]. [27] indicates that although different distinguishing coefficients usually produce different ranking results, the impact of the distinguishing coefficient on the results of GRA is very small for a multiple attribute decision-making problem (see Figure 6 in [19]). Our study follows most existing work and sets a classical grey-related parameter to 0.5.

After calculating the connection coefficient $\xi_i(k)$ between the sequence composed of weight interval number standardizing index value of every plan and reference number sequence, we need to derive an appropriate weight for the connection coefficient $\xi_i(k)$.

Step 6. *DEA-based grey related analysis.*

DEA can provide a basis for ranking alternatives based upon the set of weights that would optimize each alternative. Suppose weight w_j is unknown and crisp. The following DEA model (6) is proposed to derive the most favorable set of weights by maximizing the connection coefficient $\xi_i(k)$ between the sequence composed of the weight interval number standardizing index value of every plan and reference number sequence.

$$\begin{aligned} \theta_0 = \text{Max} \quad & \sum_{k=1}^{n} w_k \xi_0(k) \\ \text{s.t.} \quad & \sum_{k=1}^{n} w_k \xi_i(k) \le 1, \quad i = 1, 2, \ldots, m \\ & \{\text{weight normalization constraint}\} \\ & w_k \ge 0. \end{aligned} \tag{6}$$

The idea of generating objective weights using model (6) is the same as a great deal of classical DEA work such as [8,20], where weights are optimized to produce best efficiency values in a way that is beneficial to the alternative under evaluation. The reasonablity of the weights derived from DEA models similar to (6) has been validated a long while ago (see the work of Charnes et al. in 1978 [8]). Note also that our proposed DEA differs from classical DEA where the criteria are grouped into inputs and outputs due to the transformation of a "contrary index" into a positive index in Step 2. Note also that, in the proposed DEA, the weight normalization constraint is optional. A DEA model without weight normalization constraint will generate more efficient DMUs than a model with a normalization constraint. DEA model (6) is solved m times, once for each of the alternatives under evaluation. Then the ranking score for each alternative is yielded as the objective function

Table 1
Data for nuclear waste dump site selection.

	Cost	Lives	Risk	Civic
Nome	40	60	Very high	Low
Newark	100	140	Very low	Very high
Rock Sprgs	60	40	Low	High
Duquesne	60	40	Medium	Medium
Gary	70	80	Low	Very high
Yakima	70	80	High	Medium
Turkey	60	70	High	High
Wells	50	30	Medium	Medium
Anaheim	90	130	Very high	Very low
Epcot	80	120	Very low	Very low
Duckwater	80	70	Medium	Low
Santa Cruz	90	100	Very high	Very low

θ_0, $(0 \in [1, 2, \ldots, m])$. Note that $\xi_i(k)$ in the above model (6) is benefit data since we have transformed "contrary index" into positive index in Step 2. Then the DEA score given to DMU_0 can be used as the proxy measure which reflects the performance of DMU_0. All the alternatives are thus ranked according to the ranking DEA score and the optimal plan is naturally selected with the highest score.

For the weight normalization constraint, there are also many types. For example, consider the normalization constraint that confines the weight vector to the unit simplex, which leads to the following model (7),

$$\begin{aligned} \theta_0 = \text{Max} \quad & \sum_{k=1}^{n} w_k \xi_0(k) \\ & \sum_{k=1}^{n} w_k \xi_i(k) \leq 1, \quad i = 1, 2, \ldots, m \\ & \sum_{k=1}^{n} w_k = 1 \\ & w_k \geq 0. \end{aligned} \tag{7}$$

However, model (7) with this normalization always generates many efficient (non-dominated) DMUs, which are usually hard to identify, since it allows much weight flexibility. To solve the problem, a natural idea is to allow less weight restriction in (6). Moreover, model (7) might suffer from unrealistic weight distribution, i.e., the value of some weights being zero. This problem can be dealt with by executing an assurance region scheme with the decision maker's preference [28]. To adopt the assurance region scheme as commonly used in DEA issues, one might easily develop the following model (8), where the decision maker's preference over the criterion can be appended as a cone constraint.

$$\begin{aligned} \theta_0 = \text{Max} \quad & \sum_{k=1}^{n} w_k \xi_0(k) \\ & \sum_{k=1}^{n} w_k \xi_i(k) \leq 1, \quad i = 1, 2, \ldots, m \\ & w = \{w_k\} \in P, \quad k = 1, 2, \ldots, n \end{aligned} \tag{8}$$

where $P \subset E_+^m$ is a closed convex cone, and $\text{Int}\, P \neq \emptyset$.

The cone constraint in (8) defined by $w = \{w_k\} \in P$ can be actually reduced to (9) when P is a polyhedral cone given by the "intersection-form",

$$\alpha_k w_1 \leq w_k \leq \beta_k w_1, \quad k = 2, 3, \ldots, n. \tag{9}$$

Step 7: *Rank the alternatives and determine the optimal plan.*

The feasible plan X_t is called an optimal plan if $r_t = \max_{1 \leq i \leq m} r_i$.

4. Demonstration

To demonstrate the methods we present, we draw upon a hypothetical nuclear waste dump siting selection decision [29]. The multiattribute data are given in Table 1.

There are four criteria.

C1 *Cost*—cost in billions of dollars is to be minimized.
C2 *Lives lost*—expected lives lost from all exposures is to be minimized.
C3 *Risk*—the risk of catastrophe (earthquake, flood, etc.) is to be minimized.
C4 *Civic improvement*—the improvement to the local community due to construction and operation of the site is to be maximized.

Table 2
Fuzzy nuclear waste dump site data.

	Cost	Lives lost	Risk	Civic
Nome	[0.80–1.00]	[0.40–0.70]	[0.00–0.10]	[0.10–0.30]
Newark	[0.00–0.05]	[0.00–0.05]	[0.90–1.00]	[0.90–1.00]
Rock Sprgs	[0.70–0.95]	[0.70–0.90]	[0.70–0.90]	[0.70–0.90]
Duquesne	[0.50–0.85]	[0.70–0.90]	[0.40–0.60]	[0.40–0.60]
Gary	[0.40–0.60]	[0.10–0.30]	[0.70–0.90]	[0.90–1.00]
Yakima	[0.50–0.70]	[0.10–0.30]	[0.10–0.30]	[0.40–0.60]
Turkey	[0.75–0.90]	[0.20–0.40]	[0.10–0.30]	[0.70–0.90]
Wells	[0.85–0.95]	[0.85–1.00]	[0.40–0.60]	[0.40–0.60]
Anaheim	[0.00–0.30]	[0.00–0.10]	[0.00–0.10]	[0.00–0.10]
Epcot	[0.10–0.40]	[0.00–0.20]	[0.90–1.00]	[0.00–0.10]
Duckwater	[0.30–0.50]	[0.20–0.40]	[0.40–0.60]	[0.10–0.30]
Santa Cruz	[0.10–0.40]	[0.10–0.30]	[0.00–0.10]	[0.00–0.10]

Table 3
Reference number vector.

	Cost	Lives lost	Risk	Civic
Max (Min)	0.85	0.85	0.90	0.90
Max (Max)	1.00	1.00	1.00	1.00

Table 4
Distances from alternatives to reference number vector.

	Cost min	Cost max	Lives min	Lives max	Risk min	Risk max	Civic min	Civic max
Nome	0.05	0.00	0.45	0.30	0.90	0.90	0.80	0.70
Newark	0.85	0.95	0.85	0.95	0.00	0.00	0.00	0.00
Rock Sprgs	0.15	0.05	0.15	0.10	0.20	0.10	0.20	0.10
Duquesne	0.35	0.15	0.15	0.10	0.50	0.40	0.50	0.40
Gary	0.45	0.40	0.75	0.70	0.20	0.10	0.00	0.00
Yakima	0.35	0.30	0.75	0.70	0.80	0.70	0.50	0.40
Turkey	0.10	0.10	0.65	0.60	0.80	0.70	0.20	0.10
Wells	0.00	0.05	0.00	0.00	0.50	0.40	0.50	0.40
Anaheim	0.85	0.70	0.85	0.90	0.90	0.90	0.90	0.90
Epcot	0.75	0.60	0.85	0.80	0.00	0.00	0.90	0.90
Duckwater	0.55	0.50	0.65	0.60	0.50	0.40	0.80	0.70
Santa Cruz	0.75	0.60	0.75	0.70	0.90	0.90	0.90	0.90

The raw data is expressed in fuzzy intervals, as shown in Table 2. This data is standardized (fulfilling Step 3 of the prior section), with attractive values higher than lower. Each criterion is now on a common 0–1 scale where 0 represents the worst imaginable attainment on a criterion, and 1.00 the best possible attainment.

The overall value for each alternative site would be the sum product of weights time performance. DEA can be used to identify the non-dominated (efficient) alternatives by making these weights variables and optimizing the overall value for each alternative site. Conversely, fuzzy weights can be given by the decision maker or decision-making group, reflecting relative importance of criteria. We will begin by using DEA to identify a set of efficient sites.

The next step (Step 4) of the grey related method is to obtain reference number sequences based on the optimal weighted interval number value for every alternative. Since we are assuming all weights to be in the interval [0, 1] in the initial analysis, the reference number vector will be the maximum left interval value over all alternatives for each criterion, and the maximum right interval value for each criterion. Table 3 gives this vector, which reflects the range of value possibilities:

Distances are defined as the maximum between each interval value and the extremes generated in the reference number vector. Table 4 shows the calculated distances by alternative.

The maximum distance for each alternative to the ideal is identified as the largest distance calculation in each cell of Table 4. These maxima are shown in Table 5.

The minimum of the MIN column is 0.00, and the maximum of the MAX column is 0.95. Next the connection distances are calculated in Step 5 of the grey-related method (results in Table 6). Connection distances depend on the parameter ρ, which here is 0.80.

These averages were used in the grey-related method as a basis for ranking alternatives, based on the TOPSIS idea of maximizing distance from a nadir solution (worst on all criteria) and minimizing distance to an ideal solution (best on all criteria). The DEA score is calculated to determine non-dominated solutions using formulation (7) above. The DEA model is solved twelve times, each for one of the alternatives under evaluation. The formulation for an alternative Nome is:

Table 5
Maximum distances.

	Cost	Lives lost	Risk	Civic	MIN	MAX
Nome	0.05	0.45	0.90	0.80	0.05	0.90
Newark	0.95	0.95	0.00	0.00	0.00	0.95
Rock Sprgs	0.15	0.15	0.20	0.20	0.15	0.20
Duquesne	0.35	0.15	0.50	0.50	0.15	0.50
Gary	0.45	0.75	0.20	0.00	0.00	0.75
Yakima	0.35	0.75	0.80	0.50	0.35	0.80
Turkey	0.10	0.65	0.80	0.20	0.10	0.80
Wells	0.05	0.00	0.50	0.50	0.00	0.50
Anaheim	0.85	0.90	0.90	0.90	0.85	0.90
Epcot	0.75	0.85	0.00	0.90	0.00	0.90
Duckwater	0.55	0.65	0.50	0.80	0.50	0.80
Santa Cruz	0.75	0.75	0.90	0.90	0.75	0.90

Table 6
Connection distances.

	Cost	Lives lost	Risk	Civic	Rank
Nome	0.9383	0.6281	0.4578	0.4872	7
Newark	0.4444	0.4444	1.0000	1.0000	4
Rock Sprgs	0.8352	0.8352	0.7917	0.7917	1
Duquesne	0.6847	0.8352	0.6032	0.6032	5
Gary	0.6281	0.5033	0.7917	1.0000	3
Yakima	0.6847	0.5033	0.4872	0.6032	9
Turkey	0.8837	0.5390	0.4872	0.7917	6
Wells	0.9383	1.0000	0.6032	0.6032	2
Anaheim	0.4720	0.4578	0.4578	0.4578	12
Epcot	0.5033	0.4720	1.0000	0.4578	8
Duckwater	0.5802	0.5390	0.6032	0.4872	10
Santa Cruz	0.5033	0.5033	0.4578	0.4578	11

$$
\begin{aligned}
\text{Max } \theta_{\text{Nome}} = \;& 0.9383 w_{\text{cost}} + 0.6281 w_{\text{lives}} + 0.4578 w_{\text{risk}} + 0.4872 w_{\text{civic}} \\
\text{Subject to : }\;& 0.4444 w_{\text{cost}} + 0.4444 w_{\text{lives}} + 1.0000 w_{\text{risk}} + 1.0000 w_{\text{civic}} \le 1 \\
& 0.8352 w_{\text{cost}} + 0.8352 w_{\text{lives}} + 0.7917 w_{\text{risk}} + 0.7917 w_{\text{civic}} \le 1 \\
& 0.6847 w_{\text{cost}} + 0.8352 w_{\text{lives}} + 0.6032 w_{\text{risk}} + 0.6032 w_{\text{civic}} \le 1 \\
& 0.6281 w_{\text{cost}} + 0.5033 w_{\text{lives}} + 0.7917 w_{\text{risk}} + 1.0000 w_{\text{civic}} \le 1 \\
& 0.6847 w_{\text{cost}} + 0.5033 w_{\text{lives}} + 0.4872 w_{\text{risk}} + 0.6032 w_{\text{civic}} \le 1 \\
& 0.8837 w_{\text{cost}} + 0.5390 w_{\text{lives}} + 0.4872 w_{\text{risk}} + 0.7917 w_{\text{civic}} \le 1 \\
& 0.9383 w_{\text{cost}} + 1.0000 w_{\text{lives}} + 0.6032 w_{\text{risk}} + 0.6032 w_{\text{civic}} \le 1 \\
& 0.4721 w_{\text{cost}} + 0.4578 w_{\text{lives}} + 0.4578 w_{\text{risk}} + 0.4578 w_{\text{civic}} \le 1 \\
& 0.5033 w_{\text{cost}} + 0.4721 w_{\text{lives}} + 1.0000 w_{\text{risk}} + 0.4578 w_{\text{civic}} \le 1 \\
& 0.5802 w_{\text{cost}} + 0.5390 w_{\text{lives}} + 0.6032 w_{\text{risk}} + 0.4872 w_{\text{civic}} \le 1 \\
& 0.5033 w_{\text{cost}} + 0.5033 w_{\text{lives}} + 0.4578 w_{\text{risk}} + 0.4578 w_{\text{civic}} \le 1 \\
& w_i \ge 0 \quad \text{for all } i.
\end{aligned}
$$

These twelve linear programming models yielded the solutions given in Table 7.

Seven non-dominated solutions were identified, having values of 1.0 when they were optimized. This is how DEA generates relative efficiency. Were cost the only important measure, Nome or Wells would be preferred (note that the above linear programming models may have multiple optimal solutions). Were lives the only important criterion, Wells would be preferred. If risk alone was considered, Newark and Epcot would be selected. If only civic improvement was considered, Newark and Gary would be preferred. Rock Springs is non-dominated for weights emphasizing a combination of cost and risk. Turkey is non-dominated for a set of weights emphasizing cost and civic improvement. The relative efficiency of dominated solutions are given by bold entries on the diagonal of Table 8 that are less than 1.0.

Note that $\xi_i(k)$ in the above model (7) is benefit data since we have transformed the "contrary index" into a positive index in Step 2. Then the DEA score given to DMU_0 can be used as the proxy measure which reflects the performance of DMU_0. All the dominated alternatives identified could be ranked by relative DEA efficiency. However, there is no basis for ranking among the non-dominated set of alternatives.

Further information about relative weights can focus on preferred alternatives. Suppose the decision maker's preference over the criterion is reflected by the following cone-ratio form,

$$
\begin{aligned}
& 2 \ge W_{\text{lives}}/W_{\text{cost}} \ge 0.2 \\
& 5 \ge W_{\text{risk}}/W_{\text{cost}} \ge 0.5 \\
& 2 \ge W_{\text{civic}}/W_{\text{cost}} \ge 0.2,
\end{aligned}
$$

the calculated weights and assurance region (AR) DEA scores are then given in Table 8.

Table 7
DEA solutions for full range of weights.

*W*cost	1.0658	0	0.7885	0	0	0.7885	0.8986	0	0.4311	0	0.4771	0.7885
*W*lives	0	0	0	0.6547	0	0	0	1	0	0	0	0
*W*risk	0	1	0.431	0	0	0.1257	0	0	0.7616	1	0.7599	0.1257
*W*civic	0	0	0	0.5725	1	0.3056	0.2601	0	0.0468	0	0	0.3056
	Nome	Newark	Rock Sprgs	Duquesne	Gary	Yakima	Turkey	Wells	Anaheim	Epcot	Duckwater	Santa Cruz
Nome	**1.000**	0.458	0.937	0.690	0.487	0.946	0.970	0.628	0.776	0.458	0.796	0.946
Newark	0.474	**1.000**	0.782	0.864	1.000	0.782	0.659	0.444	1.000	1.000	0.972	0.782
Rock Sprgs	0.890	0.792	**1.000**	1.000	0.792	1.000	0.956	0.835	1.000	0.792	1.000	1.000
Duquesne	0.730	0.603	0.800	**0.892**	0.603	0.800	0.772	0.835	0.783	0.603	0.785	0.800
Gary	0.669	0.792	0.837	0.902	**1.000**	0.900	0.825	0.503	0.921	0.792	0.901	0.900
Yakima	0.730	0.487	0.750	0.675	0.603	**0.786**	0.772	0.503	0.694	0.487	0.697	0.786
Turkey	0.942	0.487	0.907	0.806	0.792	1.000	**1.000**	0.539	0.789	0.487	0.792	1.000
Wells	1.000	0.603	1.000	1.000	0.603	1.000	1.000	**1.000**	0.892	0.603	0.906	1.000
Anaheim	0.503	0.458	0.570	0.562	0.458	0.570	0.543	0.458	**0.574**	0.458	0.573	0.570
Epcot	0.536	1.000	0.828	0.571	0.458	0.663	0.571	0.472	1.000	**1.000**	1.000	0.663
Duckwater	0.618	0.603	0.718	0.632	0.487	0.682	0.648	0.539	0.732	0.603	**0.735**	0.682
Santa Cruz	0.536	0.458	0.594	0.592	0.458	0.594	0.571	0.503	0.587	0.458	0.588	**0.594**

Table 8
DEA solutions by model (9).

	W_{cost}	W_{lives}	W_{risk}	W_{civic}	AR score	Rank
Nome	0.5263	0.1053	0.2632	0.1053	**0.7317**	8
Newark	0.1220	0.0244	0.6098	0.2439	**0.9187**	1
Rock Sprgs	0.2703	0.5405	0.1351	0.0541	**0.8270**	5
Duquesne	0.2703	0.5405	0.1351	0.0541	**0.7506**	7
Gary	0.2703	0.0541	0.1351	0.5405	**0.8445**	4
Yakima	0.2703	0.0541	0.1351	0.5405	**0.6042**	9
Turkey	0.2703	0.0541	0.1351	0.5405	**0.7618**	6
Wells	0.2703	0.5405	0.1351	0.0541	**0.9083**	2
Anaheim	0.5263	0.1053	0.2632	0.1053	**0.4653**	12
Epcot	0.1563	0.0312	0.7813	0.0313	**0.8889**	3
Duckwater	0.1563	0.0312	0.7813	0.0313	**0.5940**	10
Santa Cruz	0.2703	0.5405	0.1351	0.0541	**0.4947**	11

The optimized weights are now all positive, indicating that four criteria all work to generate a comprehensive evaluation for the performance of alternatives. Were the AR DEA scores of identified non-dominated solutions used, the rank order of sites would be: Newark, Wells, Epcot, Gary, Rock Springs, Turkey, Duquesne, Nome, Yakima, Duckwater, Santa Cruz, and Anaheim. Table 6 had Rock Springs ranked first, followed by Wells, Gary, and Newark. Comparing the average result in the last column of Table 7, the AR DEA scores are distributed in a larger value region, which indicates its stronger diagnosing power. Moreover, the AR DEA generates realistic weights to explain the rationality of DEA scores, while the average approach fails to do this. The AR DEA scores differ from those average results because of a different weight scheme. In the AR DEA approach, the preference is reflected over the criteria indicated by the ratio interval. With the average approach, there is no readily apparent meaning to the weights.

5. Monte Carlo simulation

Compared to the average results in Table 6, DEA helps provide weight set with useful economic meanings. Using classical DEA, we yield a reference set in which non-dominated alternatives all have a value of one, as shown in Table 7. It is difficult to identify non-dominated alternatives by use of DEA efficiency values in Table 6. In order to get non-dominated solutions, we turn to cone constraint DEA by which we get the efficiency scores for all alternatives in Table 8. Based on these scores, a ranking order is finally decided, which is different from that by Table 6. In this section we employ Monte Carlo simulation to validate our approach.

Monte Carlo simulation provides a good tool to test the effect of input uncertainty over output results [19]. Monte Carlo simulation can be easily implemented by use of the simulation software Crystal Ball or @Risk, both of which are based on excel spreadsheets. It is difficult to perform classical DEA Monte Carlo simulation analysis with these products because these two softwares are not suitable for large-scale programming problems. In order to run DEA-based Monte Carlo simulation, we review the properties of the CCR DEA model as mentioned in Wu et al. [30]. From Corollary 2 in Wu et al., DMU_k, If (the rth output of DMU_k/the ith input of DMU_k) $= \max_j$ (the rth output of DMU_j/the ith input of DMU_j) then DMU_k is on the frontier.

This corollary provides a method to check the efficiency status of the DMU of interests, i.e., DMU_k. The output of the simulation model is thus defined as the maximum of the ratio of the output divided by the input, or the ratio of the some

Table 9
Statistical results of Monte Carlo simulation for 10 000 simulations.

Output name	Mean	Standard deviation	Minimum	Maximum	25th percentile	50th percentile	75th percentile	95th percentile
Epcot	0.888	0.041	0.801	1.000	0.858	0.886	0.915	0.960
Gary	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000
Newark	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000
Rock Sprgs	0.997	0.010	0.901	1.000	1.000	1.000	1.000	1.000
Turkey	0.991	0.019	0.860	1.000	0.991	1.000	1.000	1.000
Wells	1.000	0.004	0.910	1.000	1.000	1.000	1.000	1.000
Nome	0.948	0.065	0.728	1.000	0.903	0.986	1.000	1.000

Table 10
Statistical results of Monte Carlo simulation for 5000 simulations.

Output name	Mean	Standard deviation	Minimum	Maximum	25th percentile	50th percentile	75th percentile	95th percentile
Epcot	0.887	0.041	0.800	0.999	0.857	0.886	0.916	0.957
Gary	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000
Newark	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000
Rock Sprgs	0.997	0.010	0.912	1.000	1.000	1.000	1.000	1.000
Turkey	0.991	0.018	0.871	1.000	0.991	1.000	1.000	1.000
Wells	1.000	0.004	0.923	1.000	1.000	1.000	1.000	1.000
Nome	0.949	0.064	0.736	1.000	0.903	0.990	1.000	1.000

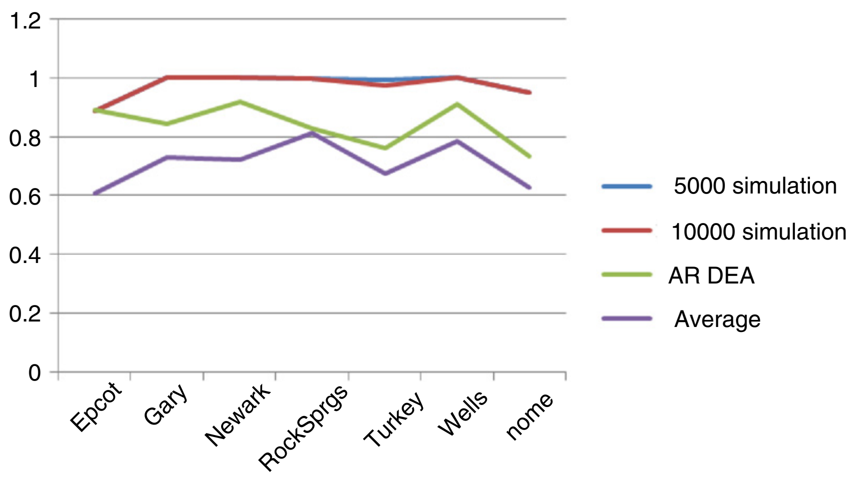

Fig. 1. Comparison of rank orders.

output summation divided by some input summation. In this way, the simulation model including all efficient branches can be well presented in the spreadsheet.

Because there are a total of seven efficient alternatives, as indicated in Table 7, we have seven outputs defined in our simulation model, each for one efficient alternative. The objective is to see the change of the efficiency value for each efficient alternative when the perturbation occurs in the fuzzy inputs of grey related analysis. The simulation inputs are defined as Uniform (current value −0.1, current value +0.1) for all fuzzy number in Table 2. If the current value is 0 or 1, then it is fixed as 0 or 1. For example, [0.80–1.00] in the first cell of Table 2 is changed to [Uniform (0.7, 0.9)–1.00] in the simulation model. Tables 9 and 10 present statistical results of Monte Carlo simulations for 10 000 and 5000 runs respectively.

Simulation results suggest three winners: Gary, Newark and Wells. These three alternatives are No. 3, No. 4, and No. 2 alternatives in Table 5, and No. 4, No. 1, and No. 2 alternatives in Table 8, and respectively. This indicates that the DEA method is more consistent with simulation results than the simple averaging method. Fig. 1 depicts the comparison of rating order for these seven alternatives.

6. Conclusions

Grey-related analysis provides a way to incorporate uncertainty into analysis. Data envelopment analysis provides a means to objectively identify non-dominated solutions. In this paper we demonstrated a way to use average functional scores obtained from grey-related analysis and DEA as a basis for rank-ordering alternatives. Our proposed DEA differs from classical DEA where the criteria are grouped into inputs and outputs since we have transformed "contrary index" into a positive index in the previous step. In the average functional approach, weight meanings are unclear. In the DEA approach, the decision maker's preference over the criteria can be appended reflected as the preference cone ratio in the model, thus realistic weights are generated to explain the rationality of DEA scores, which are finally used to select alternatives. Different versions of DEA such as super-efficiency DEA [31] may be employed to increase the discriminating power of our method. However, employment of super-efficiency DEA may fail because of infeasibility and instability problems as indicated by Thrall [32] when some inputs are close to zero which is true in our case with the connection coefficient $\xi_i(k)$ values being rounded between zero and one.

The computations we present demonstrate some of the features of the method. Results given in Table 7 show that seven of the twelve alternatives considered are nondominated. Dominated solutions can still rank high on specific metrics as shown in Table 8, but they cannot be ranked first. Tables 9 and 10 show the comparable simulation results. As shown in Table 10, Epcot did not appear nondominated with 5000 runs, although Table 9 shows that it did for the 10 000 run output. Simulation is much more direct than DEA-grey related modeling, but runs the danger of overlooking some nondominated solutions if insufficient runs are not made.

A multiattribute siting problem example was used to demonstrate our method. Monte Carlo simulation results are very robust with respect to the final rating of the alternatives by the DEA-based method. However there still are some limitations. Simulation is sometimes sensitive to input distributions. Careful design of input distributions is required. So a further consideration can be the design of the scheme to determine appropriate input distributions. Other directions are comparison studies of our approach with other standard multi-criteria methods or integrating our approach with these standard multi-criteria methods.

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